CONSIDER $y' + y = \begin{cases} 1, & x < 0 \\ 2, & x > 0 \end{cases}$, $y(0) = 3$ (FIND A CONTINUOUS SOLUTION FOR y)

FOR $x < 0$

$y' + y = 1$

$r + 1 = 0 \rightarrow r = -1$

$y_h = c_1 e^{-x}$

$y_p = A$

$y_p' = 0$

$y_p' + y_p = 0 + A = A = 1$

$y_p = 1$

$y = 1 + c_1 e^{-x}$

$\lim_{x \to 0^-} (1 + c_1 e^{-x}) = 3$

$1 + c_1 = 3$

$c_1 = 2$

FOR $x > 0$

$y' + y = 2$

$y_h = C_2 e^{-x}$

$y_p = B$

$y_p' + y_p = 0 + B = B = 2$

$y_p = 2$

$y = 2 + C_2 e^{-x}$

$\lim_{x \to 0^+} (2 + C_2 e^{-x}) = 3$

$2 + C_2 = 3$

$C_2 = 1$

$$y = \begin{cases} 1 + 2e^{-x}, & x < 0 \\ 2 + e^{-x}, & x \geq 0 \end{cases}$$

INTEGRAL OPERATOR $\mathcal{L}$ $\leftarrow$ LAPLACE TRANSFORM

IF $f : [0, \infty) \longrightarrow \mathbb{R}$

DEFINE $\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t)\,dt = \lim_{N \to \infty} \int_0^{N} e^{-st} f(t)\,dt$

(ALSO CALLED $F(s)$)

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} \cdot 1\,dt = \int_0^{\infty} e^{-st}\,dt = \lim_{N \to \infty} \int_0^{N} e^{-st}\,dt$$

$$= \lim_{N \to \infty} -\frac{1}{s} e^{-st} \Big|_0^N$$

$$= \lim_{N \to \infty} \left(-\frac{1}{s}\left(e^{-sN} - 1\right)\right)$$

$$= -\frac{1}{s}(0 - 1) \qquad s > 0$$

$$= \frac{1}{s}$$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{1\}(s) = \frac{1}{s}$

IF $f(t) = 1$, $\mathcal{L}\{f\} = \frac{1}{s}$

IF $f(t) = 1$, $F(s) = \frac{1}{s}$

$$\mathcal{L}\{cf\} = \int_0^\infty e^{-st} \cdot cf(t)\,dt = c\int_0^\infty e^{-st} f(t)\,dt = c\mathcal{L}\{f\}$$

$c \in \mathbb{R}$

$$\text{eg. } \mathcal{L}\{2\} = \mathcal{L}\{2\cdot 1\} = 2\mathcal{L}\{1\} = 2\cdot\frac{1}{s} = \frac{2}{s}$$

$$\mathcal{L}\{f+g\} = \int_0^\infty e^{-st}\cdot(f(t)+g(t))\,dt = \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt$$

$$= \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

LINEARITY OF $\mathcal{L}$

$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$ AND $\mathcal{L}\{cg\} = c\mathcal{L}\{g\}$

IF $\mathcal{L}\{f\}, \mathcal{L}\{g\}$ BOTH EXIST

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(-s+a)t}\,dt = \lim_{N\to\infty} \int_0^N e^{(-s+a)t}\,dt$$

$a \in \mathbb{R}$

$$= \lim_{N\to\infty} \frac{1}{-s+a} e^{(-s+a)t}\Big|_0^N$$

$$= \lim_{N\to\infty} \frac{1}{-s+a}\left(e^{(-s+a)N} - 1\right)$$

$$= \frac{1}{-s+a}(0-1) \qquad \begin{array}{r} -s+a<0 \\ s>a \end{array}$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}\{3e^{2t}-5e^{-4t}+7\} = 3\mathcal{L}\{e^{2t}\}-5\mathcal{L}\{e^{-4t}\}+7\mathcal{L}\{1\}$$

$$= 3\cdot\frac{1}{s-2} - 5\cdot\frac{1}{s+4} + 7\cdot\frac{1}{s}$$

$$= \frac{3}{s-2} - \frac{5}{s+4} + \frac{7}{s}$$

NOTE: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $\qquad\qquad$ $\mathcal{L}\{1\}$

SANITY CHECK: $a=0$

$$\mathcal{L}\{e^{0t}\} = \frac{1}{s-0}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \checkmark$$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \int_0^\infty te^{-st}dt = \lim_{N\to\infty}\int_0^N te^{-st}dt$$

$$\begin{array}{c|c} u & dv \\ \hline t & +e^{-st} \\ 1 & -\frac{1}{s}e^{-st} \\ 0 & \frac{1}{s^2}e^{-st} \end{array}$$

$$= \lim_{N\to\infty}\left(-\frac{1}{s}te^{-st}-\frac{1}{s^2}e^{-st}\right)\Big|_0^N$$

$$= \lim_{N\to\infty}\left(-\frac{1}{s}(Ne^{-sN}-0)-\frac{1}{s^2}(e^{-sN}-1)\right)$$

$$\lim_{N\to\infty} Ne^{-sN} = \lim_{N\to\infty}\frac{N}{e^{sN}} = \lim_{N\to\infty}\frac{1}{se^{sN}} = 0$$

$$\frac{\infty}{\infty} \qquad \frac{1}{\infty}$$

$$= -\frac{1}{s}(0) - \frac{1}{s^2}(0-1) \qquad s>0$$

$$= \frac{1}{s^2}$$

$\mathcal{L}\{mt+b\} = m\mathcal{L}\{t\} + b\mathcal{L}\{1\} = m \cdot \frac{1}{s^2} + b \cdot \frac{1}{s} = \frac{m}{s^2} + \frac{b}{s} = \frac{m+bs}{s^2}$
$m, b \in \mathbb{R}$

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}f(t)\,dt$$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ WHERE $F(s) = \mathcal{L}\{f\}$

SANITY CHECK

eg. $f(t) = 1 \longrightarrow \mathcal{L}\{f\} = \frac{1}{s} = F(s)$

$\mathcal{L}\{e^{at} \cdot f(t)\} = F(s-a)$

$\mathcal{L}\{e^{at} \cdot 1\} = \frac{1}{s-a}$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ ✓

$\mathcal{L}\{te^{at}\} = F(s-a)$ $\qquad\qquad \mathcal{L}\{t\} = \frac{1}{s^2} = F(s)$

$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$

$$\mathcal{L}\{\cos bt\} = \int_0^\infty e^{-st}\cos bt\,dt = \lim_{N\to\infty}\int_0^N e^{-st}\cos bt\,dt$$

u		dv
$\cos bt$	$+$	e^{-st}
$-b\sin bt$	$-$	$-\frac{1}{s}e^{-st}$
$-b^2\cos bt$	$+$	$\frac{1}{s^2}e^{-st}$

$$\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt + \frac{b}{s^2}e^{-st}\sin bt - \int \frac{b^2}{s^2}e^{-st}\cos bt\,dt$$

$$\left(1+\frac{b^2}{s^2}\right)\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt + \frac{b}{s^2}e^{-st}\sin bt$$

$$= \frac{s^2+b^2}{s^2}$$

$$\int e^{-st}\cos bt\,dt = -\frac{s}{s^2+b^2}e^{-st}\cos bt - \frac{b}{s^2+b^2}e^{-st}\sin bt$$